Optimization of a 3-D Thermally Asymmetric Rectangular Fin

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The non-dimensional fin length for optimum heat loss from a thermally asymmetric rectangular fin is represented as a function of the ratio of the bottom surface Biot number to the top surface Biot number, fin tip surface Biot number and the non-dimensional fin width. Optimum heat loss is taken as 98% of the maximum heat loss. For this analysis, three dimensional separation of variables method is used. Also, the relation between the ratio of the bottom surface Biot number to the top surface Biot number and the ratio of the right surface Biot number to the left surface Biot number is presented.

Key Words: 3-D Analytical Method, Optimization, Heat Loss, Biot Number

Nomenclature

- $Bi1$: Fin top Biot number, $h_1\ l/k$
- $Bi2$: Fin bottom Biot number, $h_2\ l/k$
- $Bi3$: Fin left side Biot number, $h_3\ l/k$
- $Bi4$: Fin right side Biot number, $h_4\ l/k$
- $Bi5$: Fin tip side Biot number, $h_5\ l/k$
- h_1 : Fin top heat transfer coefficient [W/m² ℃]
- h_2 : Fin bottom heat transfer coefficient [W/m² ℃]
- h_3 : Fin left side heat transfer coefficient [W/m² ℃]
- h_4 : Fin right side heat transfer coefficient [W/m² ℃]
- h_5 : Fin tip heat transfer coefficient [W/m² ℃]
- k : Thermal conductivity [W/m ℃]
- l : One half fin height at the base [m]
- L' : Fin length (base to tip) [m]
- L : Non-dimensional fin length, L'/l
- T : Fin temperature [℃]
- T_w : Fin base temperature [℃]
- T_∞ : Ambient temperature [℃]
- w' : One half fin width [m]
- w : Non-dimensional a half fin width, w'/l
- x' : Length directional variable [m]
- x : Non-dimensional length directional variable, x'/l
- y' : Height directional variable [m]
- y : Non-dimensional height directional variable, y'/l
- z' : Width directional variable [m]
- z : Non-dimensional width directional variable, z'/l
- θ_0 : Adjusted temperature, $(T_w - T_\infty)$
- θ : Non-dimensional temperature, $(T - T_\infty)/(T_w - T_\infty)$
- λ_n : Eigenvalues ($n=1, 2, 3, \cdots$)
- μ_m : Eigenvalues ($m=1, 2, 3, \cdots$)
- ρ_{nm} : Eigenvalues ($\sqrt{\lambda_n^2+\mu_m^2}$)

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1. Introduction

Fins are widely used to enhance the rate of heat transfer to a surrounding fluid in many engineering applications such as the cooling of combustion engines, many kind of heat exchangers, air craft and so on. Optimization of various shapes of fins have been studied. For example, Georgiou (1998), Gerencser and Razan (1995) and Ledezma et al. (1996) have discussed pin fins, Look and Kang (1992) were concerned with rectangular while Georgiou (1998) examined trapezoidal. Also Ullmann and Kalman

(1989) presented annular fin and Yu and Chen (1999) and Zubair et al. (1996) researched circular fins. Usually most of the studies on the fin assume that the heat transfer coefficients for all surfaces of the fin are equal. For thermally asymmetric study, Look and Kang (1992) discussed 2-D rectangular fin with three different heat convection coefficients. There are also some papers which deal with asymmetric profile fin. For example, Kang (1997) have discussed asymmetric trapezoidal fin while Shah (1971) presented Table which shows asymmetric straight rectangular plate-fin. All these papers have been studied by one- or two-dimensional analysis. But no literature seems to be available which presents optimization of a rectangular fin with unequal heat transfer coefficients by using three-dimensional analysis.

Actually the thermal condition of the fin in the margin of fin arrays will be asymmetric. This study produced an optimization procedure for the heat loss from a thermally asymmetric rectangular fin using three-dimensional separation of variables method. In this study the upper surface Biot number, Bi1, is equal to or larger than the bottom surface Biot number, Bi2. The left surface Biot number, Bi3, is equal to or larger than the right surface Biot number, Bi4, and Bi5, at the fin tip, has various values even though these situations are somewhat artificial. The non-dimensional fin length for optimum heat loss is investigated as a function of the non-dimensional fin width, fin tip surface Biot number and the ratio of the bottom surface Biot number to the top surface Biot number (Bi2/Bi1). The optimum heat loss is taken as 98% of the maximum heat loss for given conditions by showing the ratio of heat loss to the maximum heat loss with the variation of the non-dimensional fin length. Further, for arbitrary thermally asymmetric circumstances, the relation between Bi2/Bi1 and Bi4/Bi3 for the same fin length for optimum heat loss is presented. For simplicity, the root temperature and the thermal conductivity of the fin's material are assumed constant as well as steady-state.

2. Three-Dimensional Analysis

When fins are arrayed as shown in Fig. 1(a), the fin on the corner can be considered to be under thermally asymmetric condition. In this case the general rectangular (not square) fin on the corner can be approximated to the fin which is shown in Fig. 1(b).

Three-dimensional governing differential equation under steady state for the fin in Fig. 1(b) is

$$\frac{\partial^2 \theta}{\partial x^2}+\frac{\partial^2 \theta}{\partial y^2}+\frac{\partial^2 \theta}{\partial z^2}=0 \qquad (1)$$

Six boundary conditions are required to solve the Eq. (1). These boundary conditions are

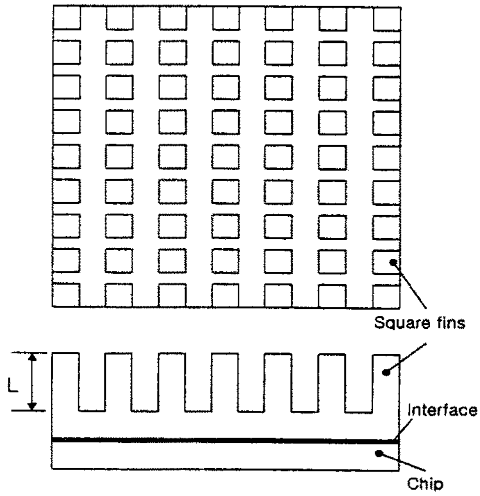

Fig. 1(a) Geometry of fin array

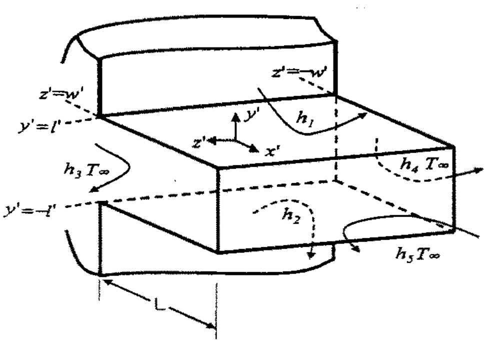

Fig. 1(b) Geometry of a thermally asymmetric rectangular fin shown as Eqs. (2) through (7).

$$\theta=1 \text{ at } x=0 \tag{2}$$

$$\frac{\partial\theta}{\partial x}+Bi5\cdot\theta=0 \text{ at } x=L \tag{3}$$

$$\frac{\partial\theta}{\partial y}+Bi1\cdot\theta=0 \text{ at } y=1 \tag{4}$$

$$\frac{\partial\theta}{\partial y}-Bi2\cdot\theta=0 \text{ at } y=-1 \tag{5}$$

$$\frac{\partial\theta}{\partial z}+Bi3\cdot\theta=0 \text{ at } z=w \tag{6}$$

$$\frac{\partial\theta}{\partial z}-Bi4\cdot\theta=0 \text{ at } z=-w \tag{7}$$

The solution for the temperature distribution $\theta(x, y, z)$ within the thermally asymmetric rectangular fin obtained using separation of variables method with Eqs. (2) through (5) is

$$\theta(x, y, z)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}N_{nm}\cdot f(x)\cdot f(y)\cdot f(z) \tag{8}$$

where

$$N_{nm}=\frac{4\sin\lambda_n\cdot\sin(u_m\cdot w)}{f_n\cdot g_m} \tag{9}$$

$$f(x)=\cosh(\rho_{nm}\cdot x)-C_{nm}\cdot\sinh(\rho_{nm}\cdot x) \tag{10}$$

$$C_{nm}=\frac{\rho_{nm}\cdot\tanh(\rho_{nm}\cdot L)+Bi5}{\rho_{nm}+Bi5\cdot\tanh(\rho_{nm}\cdot L)} \tag{11}$$

$$\rho_{nm}=\sqrt{(\lambda_n^2+\mu_m^2)} \tag{12}$$

$$f(y)=\cos(\lambda_n\cdot y)+A_n\cdot\sin(\lambda_n\cdot y) \tag{13}$$

$$A_n=\frac{\lambda_n\cdot\tan(\lambda_n)-Bi1}{\lambda_n+Bi1\cdot\tan(\lambda_n)} \tag{14}$$

$$f(z)=\cos(\mu_m\cdot z)+B_m\cdot\sin(\mu_m\cdot z) \tag{15}$$

$$B_m=\frac{\mu_m\cdot\tan(\mu_m\cdot w)-Bi3}{\mu_m+Bi3\cdot\tan(\mu_m\cdot w)} \tag{16}$$

$$f_n=\lambda_n+\frac{1}{2}\sin(2\lambda_n)+A_n^2\cdot\left\{\lambda_n-\frac{1}{2}\sin(2\lambda_n)\right\} \tag{17}$$

$$g_m=\mu_m w+\frac{1}{2}\sin(\mu_m\cdot w)+B_m^2\cdot\left\{\mu_m\cdot w-\frac{1}{2}\sin(\mu_m\cdot w)\right\} \tag{18}$$

The eigenvalues λ_n can be obtained from Eq. (19) which comes from Eq. (4) and Eq. (5).

$$\frac{\lambda_n\cdot\tan(\lambda_n)-Bi1}{\lambda_n+Bi1\cdot\tan(\lambda_n)}=\frac{Bi2-\lambda_n\cdot\tan(\lambda_n)}{\lambda_n+Bi2\cdot\tan(\lambda_n)} \tag{19}$$

Similarly the eigenvalues μ_m can be obtained from Eq. (20) which comes from Eq. (6) and Eq. (7).

$$\frac{\mu_m\cdot\tan(\mu_m\cdot w)-Bi3}{\mu_m+Bi3\cdot\tan(\mu_m\cdot w)}=\frac{Bi4-\mu_m\cdot\tan(\mu_m\cdot w)}{\mu_m+Bi4\cdot\tan(\mu_m\cdot w)} \tag{20}$$

The first eigenvalues of λ_n and μ_m are obtained by incremental search method and the rest of eigenvalues of λ_n and μ_m are calculated using forced analytic method. By applying Eq. (8) to Fourier's law, the heat loss rate conducted into the fin through the fin base is given by

$$Q=4k\cdot l\cdot\theta_0\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}N_{nm}\cdot\rho_{nm}\cdot C_{nm}\cdot\frac{\sin(\lambda_n)}{\lambda_n}\cdot\frac{\sin(\mu_m\cdot w)}{\mu_m} \tag{21}$$

where $\theta_0=T_w-T_\infty$

In order to obtain the limiting value of heat loss with respect to the non-dimensional fin length, Eq. (21) can be differentiated with respect to L and set to be 0 and it is shown by Eq. (22).

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}N_{nm}\cdot\rho_{nm}\cdot fd_{nm}\cdot\frac{\sin(\lambda_n)}{\lambda_n}\cdot\frac{\sin(\mu_m\cdot w)}{\mu_m}\cdot\sec h^2(\rho_{nm}\cdot L)=0 \tag{22}$$

where, $fd_{nm}=\dfrac{\rho_{nm}(\rho^2_{nm}-Bi5^2)}{\{\rho_{nm}+Bi5\cdot\tanh(\rho_{nm}\cdot L)\}^2}$

Equation (22) will be satisfied as L approaches infinity; then $\tanh(\rho_{nm}L)\rightarrow 1$ and $C_{nm}\rightarrow 1$ (see Eq. (11)). So under our usual circumstances (i.e. $Bi<0.1$), the maximum heat loss can be expressed by Eq. (23).

$$Q_{\max}=4k\cdot l\cdot\theta_0\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}N_{nm}\cdot\rho_{nm}\cdot\frac{\sin(\lambda_n)}{\lambda_n}\cdot\frac{\sin(\mu_m\cdot w)}{\mu_m} \tag{23}$$

Two hundreds of λ_n and two hundreds of μ_m (i.e. $200\times200=40{,}000$) are used to calculate the heat loss and the maximum heat loss.

3. Results and Discussions

Table 1 lists the ratio of heat loss to the maximum heat loss with the variation of non-dimensional fin length for thermally symmetric condition. This table shows that the ratio approaches 100% more quickly as Biot number increases and non-dimensional fin width de-

Table 1 The ratio of heat loss to the maximum heat loss for a thermally symmetric rectangular fin

L	Q / Q_{max} (as $L \to \infty$) (%)			
	Bi = 0.01		Bi = 0.1	
	w = 0.5	w = 10	w = 0.5	w = 10
1	22.69	19.81	62.29	56.92
2	38.33	29.63	85.42	74.95
4	63.51	47.36	98.21	92.46
8	89.41	73.20	99.98	99.41
12	97.24	87.43	100.00	99.96
16	99.30	94.36	100.00	100.00
20	99.82	97.52	100.00	100.00

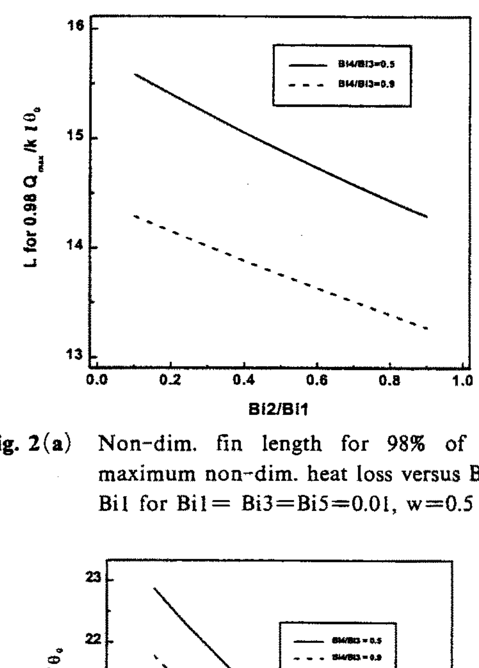

Fig. 2(a) Non-dim. fin length for 98% of the maximum non-dim. heat loss versus Bi2/Bi1 for Bi1= Bi3=Bi5=0.01, w=0.5

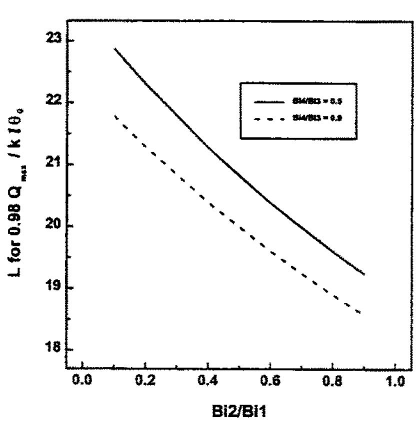

Fig. 2(b) Non-dim. fin length for 98% of the maximum non-dim. heat loss versus Bi2/Bi1 for Bi1= Bi3=Bi5=0.01, w=2

creases. It can be noted that the fin length must be increased over twice to get the last 2% of the ratio. So the non-dimensional fin length for 98% of the maximum heat loss will be considered as the optimum non-dimensional fin length in this study.

Figure 2(a) presents the non-dimensional fin length for 98% of the maximum non-dimensional heat loss versus Bi2/Bi1 for Bi1=Bi3=Bi5=0.01, w=0.5. It shows that the optimum non-dimensional fin length increases almost linearly for both values of Bi4/Bi3 as Bi2/Bi1 decreases. The same description but for w=2 case is shown in Fig. 2(b). Comparing this figure to Fig. 2(a), the variation trend is almost the same but the required fin length for 98% of the maximum heat loss lengthens and it decreases more rapidly as Bi2/Bi1 increases. It can also be noted from two figures that the magnitude difference for the optimum non-dimensional fin length between Bi4/Bi3=0.5 and Bi4/Bi3=0.9 slightly decreases as Bi2/Bi1 increases.

Figures 3(a), (b) show the variation of the optimum non-dimensional fin length as a function of fin tip Biot number. The optimum non-dimensional fin length decreases almost linearly for w=0.5 while it decreases slightly curved for w=2 as fin tip Biot number increases. It must be noted that the magnitude between the optimum non-dimensional fin length for Bi2/Bi1=0.9, Bi4/Bi3=0.7 and that for Bi2/Bi1=0.7, Bi4/Bi3=0.9 is reversed as the non-dimensional fin width changes from 0. 5 to 2. Especially from Fig. 3(b), it can be shown that the effect of the values Bi2/Bi1 and Bi4/Bi3 on the optimum non-dimensional fin length decreases as fin tip Biot number increases.

Figure 4 presents the optimum non-dimensional fin length versus the non-dimensional fin width when Bi1, Bi3 and Bi5 are fixed as 0.01. This figure shows that the optimum non-dimensional fin length increases rapidly as the non-dimensional width increases from 0.1 to 2, and then the increasing rate of the optimum non-dimensional fin length is slow down as the non-dimensional width increases. The optimum non-dimensional fin length for Bi2/Bi1=1.0 and

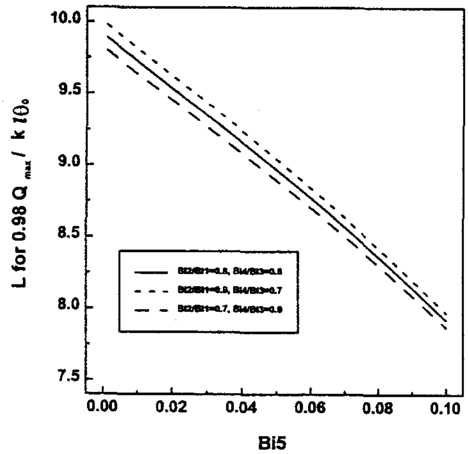

Fig. 3(a) Non-dim. fin length for 98% of the maximum non-dim. heat loss versus Bi5 for w=0.5 and Bi1=Bi3=0.02

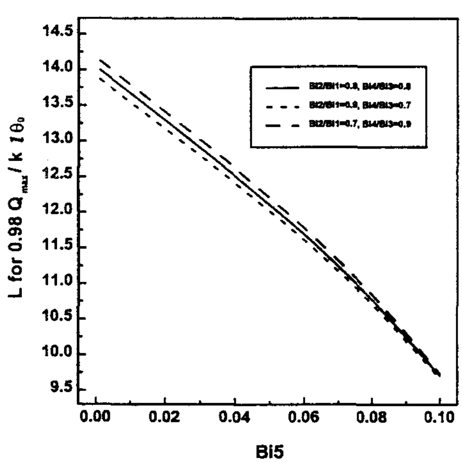

Fig. 3(b) Non-dim. fin length for 98% of the maximum non-dim. heat loss versus Bi5 for w=2 and Bi1=Bi3=0.02

Bi4/Bi3=0.8 is larger than that for Bi2/Bi1=0.8 and Bi4/Bi3=1.0 at the same width and the difference between these two values can be noticed about over w=2.

The variations of the optimum non-dimensional fin length as a function of the non-dimensional fin width for several values of fin tip Biot number in case of Bi1=Bi3=0.01, Bi2/Bi1=Bi4/Bi3=0.9 are described in Fig. 5. The trend of variation for the optimum non-dimensional fin length is somewhat similar to that in Fig. 4. It also shows the optimum fin length decreases as fin tip Biot number increases for the same width. From last two figures, it can be guessed the effect of width on the optimum fin length for given Biot numbers seems to be independent at large values of fin width.

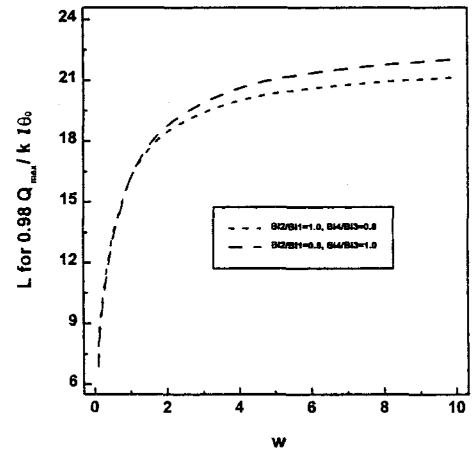

Fig. 4 Non-dim. fin length for 98% of the maximum heat loss versus w for Bi1=Bi3=Bi5=0.01

Kang et al. (2001) presented the performance of the rectangular fin under symmetric condition using three-dimensional analytic method. To validate the numerical results of the present work, relative error of heat loss from a rectangular fin between symmetric analysis and asymmetric (but all Biot number set to be equal) analysis is listed in Table 2. The relative errors shown in Table 2 are less than 1×10^{-3}. So the results in this study can be expected to be accurate even though the equations for asymmetric condition are somewhat complicate.

Figure 6 illustrates the relation between Bi2/Bi1 and Bi4/Bi3 for the same L for 98% of the maximum heat loss when Bi1, Bi3 and Bi5 are fixed as 0.01. This figure shows that the value of Bi4/Bi3 decreases linearly as Bi2/Bi1 increases to satisfy the same optimum fin length. It also shows the variation slope for w=0.5 is larger than that for w=0.1 and it can be explained physically that the effect of the ratio of the bottom surface Biot number to the top surface Biot number on the ratio of the right surface Biot number to the left surface Biot number becomes important as fin

Table 2 Relative error of heat loss between symmetric analysis and asymmetric analysis (but all Biot number set to be equal) for L=10

w	$((Q_{asy}-Q_{sym})$ / Q_{sym} (%)	
	Bi = 0.01	Bi = 0.1
0.1	0	0
1	2×10^{-4}	0
2	3.6×10^{-4}	3×10^{-5}
4	4×10^{-4}	5×10^{-5}
6	4.2×10^{-4}	1×10^{-5}
8	4.7×10^{-4}	5×10^{-5}
10	4.7×10^{-4}	7×10^{-5}

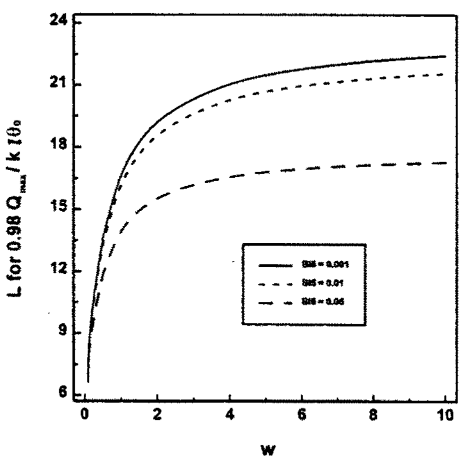

Fig. 5 Non-dim. fin length for 98% of the maximum heat loss versus w for Bi2/Bi1=Bi4/Bi3=0.9, Bi1=Bi3=0.01

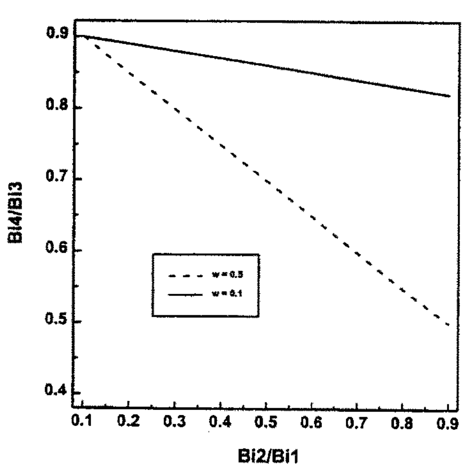

Fig. 6 The relation between Bi2/Bi1 and Bi4/Bi3 for the same L for 98% of the maximum heat loss in case of Bi1=Bi3=Bi5=0.01

width increases.

4. Conclusions

The following conclusions can be made from the results.

(1) The fin length for optimum heat loss decreases as Bi2/Bi1 increases or as Bi5 increases for an arbitrary thermally asymmetric condition.

(2) The effect of the non-dimensional width on the fin length for optimum heat loss is remarkable when the non-dimensional width is narrow (i.e. approximately $w \leq 4$).

(3) The optimum fin length decreases as fin tip Biot number increases for the same width.

(4) For the same fin length for optimum heat loss in an arbitrary thermally asymmetric case, Bi4/Bi3 decreases linearly as Bi2/Bi1 increases.

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